2 Trigonometry - 1

Let's Study

- Trigonometric functions with the help of unit circle
- Extensions of trigonometric functions to any angle
- Range and Signs of trigonometric functions in different quadrants
- Fundamental Identities and Periodicity of trigonometric functions
- Domain, Range and Graph of each trigonometric function
- Polar Co-ordinates

2.1 Introduction

Trigonometry is a branch of Mathematics that deals with the relation between sides and angles of triangles. The word 'trigonometry' is derived from the Greek words 'trigonon' and 'metron'. It means measuring the sides of triangles. Greek Mathematicians used trigonometric ratios to determine unknown distances. The Egyptians used a primitive form of trigonometry for building pyramids in the second millennium BC. Greek astronomer Hipparches (190-120 BC) formulated the general principles of trigonometry and he is known as the founder of the trigonometry.

We are familiar with trigonometric ratios of acute angles a in right angled triangle. We have introduced the concept of directed angle having any measure, in the previous chapter. We shall now extend the definitions of trigonometric ratios to angles of any measure in terms of co-ordinates of points on the standard circle.

Let's Recall

We have studied that, in a right angled triangle if measure of an acute angle is 'θ', then

$$\sin\theta = \frac{\text{opposite side}}{\text{hypoteneous}}, \quad \cos\theta = \frac{\text{adjacent side}}{\text{hypoteneous}},$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}.$$ (see fig 2.1 (a))

$$\text{Also, } \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta},$$

$$\cot\theta = \frac{1}{\tan\theta}.$$

Let's Learn

2.1.1 Trigonometric functions with the help of a circle:

Trigonometric ratios of any angle

We have studied that in right angled Δ ABC, 'θ' is an acute angle

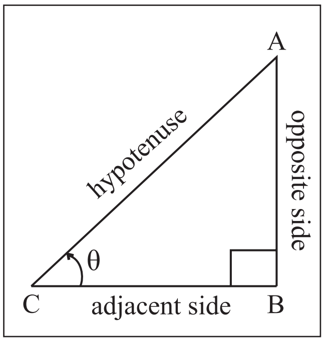

Fig. 2.1(a)

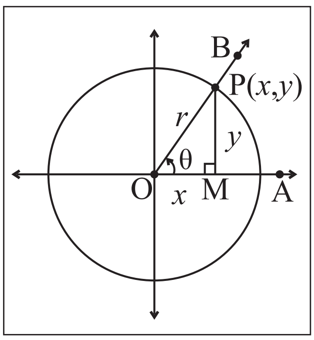

Fig. 2.1(b)

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypoteneous}} = \frac{BC}{AC}$$

$$\sin\theta = \frac{\text{opposite side}}{\text{hypoteneous}} = \frac{AB}{AC}$$

We will now extend this definition to any angle θ, consider θ as directed angle,

Let 'θ' be an acute angle. [See fig. 2.1 (b)]

consider a circle of radius 'r' with centre at origin 'O' of the co-ordinate system.

OA is the initial ray of angle θ,

OB is its terminal ray.

$P(x,y)$ is a point on the circle and on ray OB.

Draw PM $\perp^{er}$ to OA.

$\therefore$ OM = x, PM = y and OP = r.

using Δ PMO we get,

$$\cos\theta = \frac{OM}{OP} = \frac{x}{r}, \quad \sin\theta = \frac{PM}{OP} = \frac{y}{r},$$

then we define

$$\cos\theta = \frac{x}{r} = \frac{x-\text{co-ordinate of P}}{\text{Distance of P from origin}}$$

$$\sin\theta = \frac{y}{r} = \frac{y-\text{co-ordinate of P}}{\text{Distance of P from origin}}$$

and $r^2 = x^2 + y^2$

Hence, $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$

For every angle 'θ', there is corresponding unique point $P(x,y)$ on the circle, which is on the terminal ray of 'θ', so trignometric ratio's of θ are also trignometric functions of 'θ'.

Note that : 1) Trignometric ratios / functions are independent of radius 'r'.

2) Trignometric ratios of coterminal angles are same.

We consider the circle with center at origin and radius r. Let $P(x,y)$ be the point on the circle with $m\angle MOP = \theta$

Since P lies on the circle, $OP = r$

$\therefore \quad \sqrt{x^2+y^2} = r$

The definitions of sinθ, cosθ and tanθ can now be extended for $\theta = 0°$ and $90° \le \theta \le 360°$. We will also define secθ, cosecθ and cotθ.

Every angle θ, $0° \le \theta \le 360°$, determines a unique point P on the circle so that OP makes angle θ with X-axis.

The pair (x,y) of co-ordinates of P is uniquely determined by θ. Thus $x = r\cos\theta$, $y = r\sin\theta$ are functions of θ.

Note :

1) If $P(x, y)$ lies on the unit circle then $\cos\theta = x$ and $\sin\theta = y$. $\therefore$ $P(x, y) \equiv P(\cos\theta, \sin\theta)$

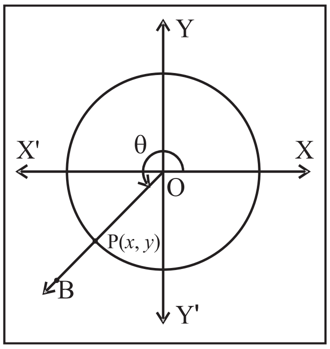

Fig. 2.2

2) The trigonometric functions do not depend on the position of the point on the terminal arm but they depend on measure of the angle.

Point $P(x,y)$ is on the circle of radius r and $Q(x',y')$ is on the unit circle.

Considering results on similar triangles.

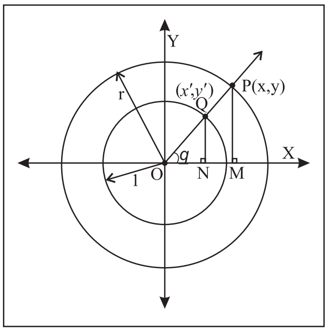

Fig. 2.3

$\sin\theta = \frac{y}{r} = \frac{y'}{1}$,

$\therefore y = r\sin\theta$

$y^1 = \sin\theta$ and

$\cos\theta = \frac{x}{r} = \frac{x'}{1}$, $x = r\cos\theta$ $x^1 = \cos\theta$

2.1.2 Signs of trigonometric functions in different quadrants :

Trigonometric functions have positive or negative values depending on the quadrant in which the point $P(x, y)$ lies. Let us find signs of trigonometric ratios in different quadrants. If the terminal arm of an angle θ intersects the unit circle in the point $P(x, y)$, then $\cos\theta = x$.

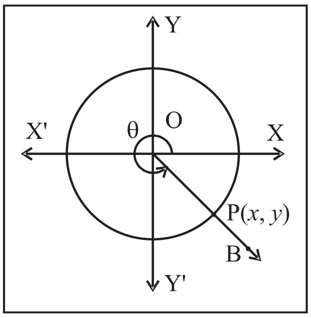

Fig. 2.4

$\sin\theta = y$ and $\tan\theta = \dfrac{y}{x}$. The values of x and y are positive or negative depending on the quadrant in which P lies.

1) In the first quadrant ($0 < \theta < \dfrac{\pi}{2}$), both x and y are positive, hence

$\cos\theta = x$ is positive

$\sin\theta = y$ is positive

$\tan\theta = \dfrac{y}{x}$ is positive

Hence all trigonometric functions of θ are positive in the first quadrant.

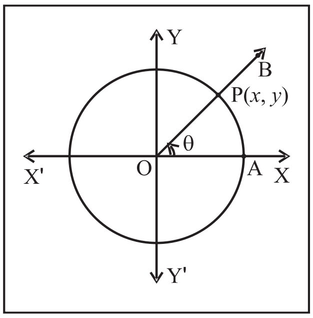

Fig. 2.5

2) In the second quadrant ($\dfrac{\pi}{2} < \theta < \pi$), y is positive and x is negative, hence

$\cos\theta = y$ is positive

$\sin\theta = x$ is is negative

$\tan\theta = \dfrac{y}{x}$ is negative

Hence only $\sin\theta$ is positive, $\cos\theta$ and $\tan\theta$ are negative for θ in the second quadrant.

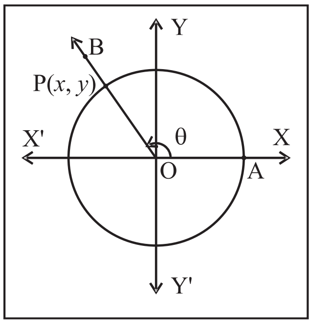

Fig. 2.6

3) In the third quadrant ($\pi < \theta < \dfrac{3\pi}{2}$), both x and y are negative, hence

$\cos\theta = y$ is negative

$\sin\theta = x$ is negative

$\tan\theta = \dfrac{y}{x}$ is positive

Hence only $\tan\theta$ is positive $\sin\theta$ and $\cos\theta$ are negative for θ in the third quadrant.

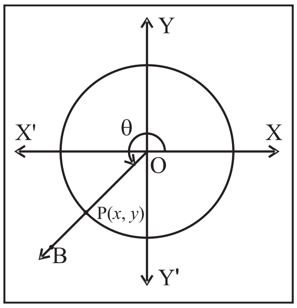

Fig. 2.7

4) In the fourth quadrant ($\dfrac{3\pi}{2} < \theta < 2\pi$), x is positive and y is negative, hence

$\sin\theta = y$ is negative

$\cos\theta = x$ is positive

$\tan\theta = \dfrac{y}{x}$ is negative

Hence only $\cos\theta$ is positive; $\sin\theta$ *and* $\tan\theta$ are negative for θ in the fourth quadrant.

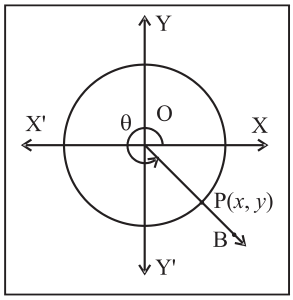

Fig. 2.8

You can check $\sin\theta$ & $\operatorname{cosec}\theta$, have the same sign, $\cos\theta$ & $\sec\theta$ have the same sign and simillarly $\tan\theta$ & $\cot\theta$ have the same sign, when they exists.

Remark: Signs of $\operatorname{cosec}\theta$, $\sec\theta$ *and* $\cot\theta$ are same as signs of $\sin\theta$, $\cos\theta$ *and* $\tan\theta$ respectively.

2.1.3 Range of *cosθ* and *sinθ* : $P(x, y)$ is point on the unit circle. $m\angle AOB = \theta$. $OP = 1$

$\therefore\ x^2 + y^2 = 1$

$\therefore\ x^2 \le 1$ and $y^2 \le 1$

$\therefore\ -1 \le x \le 1$ and $-1 \le y \le 1$

$\therefore\ -1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$

SOLVED EXAMPLE

Ex.1. Find the signs of the following :

i) sin 300° ii) cos 400° iii) cot (–206°)

Solution:

(For given θ, we need to find coterminal angle which lies between 0° and 360°)

i) $270° < 300° < 360°$

$\therefore$ 300° angle lies in the fourth quadrant.

$\therefore$ sin 300° is negative.

ii) $400° = 360° + 40°$

$\therefore$ 400° and 40° are co-terminal angles (hence their trigonometric ratios are same)

Since 40° lies in the first quadrant, 400° also lies in the first quadrant.

$\therefore$ cos 400° is positive.

iii) $-206° = -360° + 154°$

154° and –206° are coterminal angles. Since 154° lies in the second quadrant, therefore cot (–206°) is negative.

2.1.4 Trigonometric Functions of specific angles

1) Angle of measure 0° : Let $m\angle XOP = 0°$. Its terminal arm intersects unit circle in $P(1,0)$. Hence $x = 1$ and $y = 0$.

We have defined,

$\sin\theta = y$, $\cos\theta = x$

and $\tan\theta = \dfrac{y}{x}$

$\therefore$ $\sin 0° = 0$, $\cos 0° = 1$,

and $\tan 0° = \dfrac{0}{1} = 0$

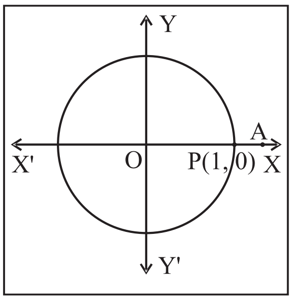

Fig. 2.9

cosec0° is not defined as $y = 0$, sec0° = 1 and cot0° is not defined as $y = 0$

2) Angle of measure 90° *or* $\left(\dfrac{\pi}{2}\right)^c$: Let $m\angle XOP = 90°$. Its terminal arm intersects unit circle in $P(0,1)$.

Hence $x = 0$ and $y = 1$

$\therefore$ $\sin 90° = y = 1$

$\cos 90° = x = 0$

tan90° is not defined as cos90° = 0

$\text{cosec}90° = \dfrac{1}{y} = \dfrac{1}{1} = 1$

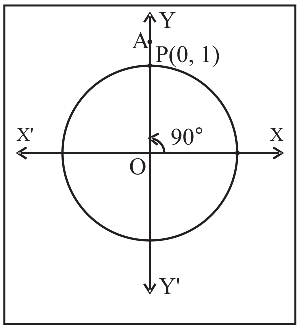

Fig. 2.10

sec90° is not defined as $x = 0$

$\cot 90° = \dfrac{x}{y} = \dfrac{0}{1} = 0$

(Activity) :

Find trigonometric functions of angles 180°, 270°.

3) Angle of measure 360° or $(2\pi)^c$: Since 360° and 0° are co-terminal angles, trigonometric functions of 360° are same as those of 0°.

4) Angle of measure 120° or $\left(\dfrac{2\pi}{3}\right)^c$:

Let $m\angle XOP = 120°$. Its terminal arm intersects unit circle in $P(x, y)$.

Draw PQ perpendicular to the X-axis

$\therefore$ Δ OPQ is 30° – 60° – 90° triangle.

$\therefore$ $OQ = \dfrac{1}{2}$ and $PQ = \dfrac{\sqrt{3}}{2}$ and $OP = 1$

As P lies in the second quadrant, $x = -\dfrac{1}{2}$ and $y = \dfrac{\sqrt{3}}{2}$

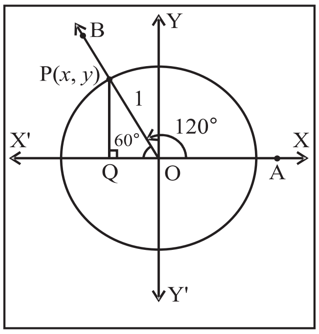

Fig. 2.11

$\therefore \ \sin 120° = y = \frac{\sqrt{3}}{2}$

$\cos 120° = x = -\frac{1}{2}$

$\tan 120° = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$

$\text{cosec}120° = \frac{1}{y} = \frac{2}{\sqrt{3}}$

$\sec 120° = \frac{1}{x} = -2$

$\cot 120° = \frac{x}{y} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$

5) Angle of measure 225° or $\left(\frac{5\pi^c}{4}\right)$

Let $m\angle XOP = 225°$. Its terminal arm intersects unit circle in $P(x,y)$. Draw PQ perpendicular to the X-axis at Q.

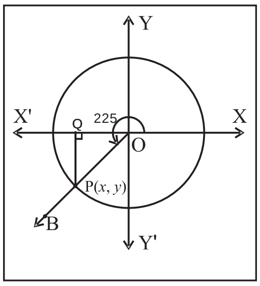

Fig. 2.12

$\therefore$ Δ OPQ 45° – 45° – 90° triangle.

$\therefore$ OQ $= \frac{1}{\sqrt{2}}$ and $PQ = \frac{1}{\sqrt{2}}$ and $OP = 1$

As P lies in the third quadrant, $x = -\frac{1}{\sqrt{2}}$ and $y = -\frac{1}{\sqrt{2}}$

$\therefore \ \sin 225° = y = -\frac{1}{\sqrt{2}}$

$\cos 225° = x = -\frac{1}{\sqrt{2}}$

$\tan 225° = \frac{y}{x} = 1$

$\text{cosec}225° = \frac{1}{y} = -\sqrt{2}$

$\sec 225° = \frac{1}{x} = -\sqrt{2}$

$\cot 225° = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1$

2.1.3 Trigonometric functions of negative angles:

Let $P(x, y)$ be any point on the unit circle with center at the origin such that $\angle AOP = \theta$.

If $\angle AOQ = -\theta$. then the co-ordinates of Q will be $(x, -y)$.

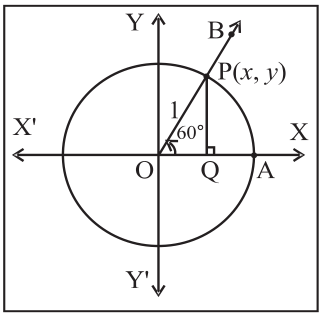

Fig. 2.13

By definition

$\sin\theta = y$ and $\sin(-\theta) = -y$

$\cos\theta = x$ and $\cos(-\theta) = x$

Therefore $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\mathbf{tan}\,\theta$$

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\mathbf{cot}\,\theta$$

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \mathbf{sec}\,\theta$$

$$\text{cosec}(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\mathbf{cosec}\,\theta$$

6) Angle of measure – 60° or $-\frac{\pi}{3}$:

Let $m\angle XOP = -60°$.

Its terminal arm intersects unit circle in $P(x, y)$.

Draw PQ perpendicular to the X-axis .

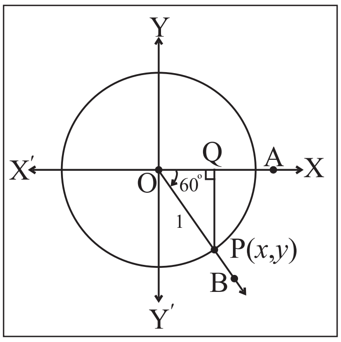

Fig. 2.14

$\therefore$ Δ OPQ is 30° – 60° – 90° triangle.

$OQ = \frac{1}{2}$ and $PQ = \frac{\sqrt{3}}{2}$ and OP = 1

As P lies in the fourth quadrant, $x = \frac{1}{2}$ and $y = -\frac{\sqrt{3}}{2}$

$$\therefore \sin(-60°) = y = -\frac{\sqrt{3}}{2}$$

$$\cos(-60°) = x = \frac{1}{2}$$

$$\tan(-60°) = \frac{y}{x} = \frac{\frac{-\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

$$\operatorname{cosec}(-60°) = \frac{1}{y} = -\frac{2}{\sqrt{3}}$$

$$\sec(-60°) = \frac{1}{x} = 2$$

$$\cot(-60°) = \frac{x}{y} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

Note : Angles –60° and 300° are co-terminal angles therefore values of their trigonometric functions are same.

The trigonometric functions of 0°, 30°, 45°, 60°, 90° are tabulated in the following table.

Angles \ Trig. Fun.	sinθ	cosθ
360° = 0^c	0	1
$30° = \frac{\pi^c}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$45° = \frac{\pi^c}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$60° = \frac{\pi^c}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$90° = \frac{\pi^c}{2}$	1	0
180° = π	0	–1
$270° = \frac{3\pi}{2}$	–1	0

(Activity) :

Find trigonometric functions of angles 150°, 210°, 330°, – 45°, – 120°, $-\frac{3\pi}{4}$ and complete the table.

Trig. Fun. \ θ Angle	sin θ	cos θ	tan θ	cosecθ	sec θ	cot θ
150°						
210°						
330°						
–45°						
–120°						
$-\frac{3\pi}{4}$						

SOLVED EXAMPLES

Ex.1 For $\theta = 30°$, Verify that $\sin 2\theta = 2\sin\theta\cos\theta$

Solution: Given $\theta = 30°$ $\therefore$ $2\theta = 60^0$

$$\sin\theta = \sin 30° = \frac{1}{2}$$

$$\cos\theta = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\sin 2\theta = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\text{L.H.S.} = 2\sin\theta\cos\theta = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2} = \sin 2\theta = \text{R.H.S.}$$

Ex.2 Evaluate the following :

i) $\cos 30° \times \cos 60° + \sin 30° \times \sin 60°$

ii) $4\cos^3 45° - 3\cos 45° + \sin 45°$

iii) $\sin^2 0 + \sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{2}$

iv) $\sin\pi + 2\cos\pi + 3\sin\frac{3\pi}{2} + 4\cos\frac{3\pi}{2} - 5\sec\pi - 6\,\text{cosec}\,\frac{3\pi}{2}$

Solution :

i) $\cos 30° \times \cos 60° + \sin 30° \times \sin 60°$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

ii) $4\cos^3 45° - 3\cos 45° + \sin 45°$

$$= 4\left(\frac{1}{\sqrt{2}}\right)^3 - 3\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= 4\,\frac{1}{2\sqrt{2}} - \frac{2}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} = 0$$

iii) $\sin^2 0 + \sin^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{2}$

$$= (0)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (1)^2$$

$$= 0 + \frac{1}{4} + \frac{3}{4} + 1 = 2$$

iv) $\sin\pi + 2\cos\pi + 3\sin\frac{3\pi}{2} + 4\cos\frac{3\pi}{2} - 5\sec\pi - 6\,\text{cosec}\,\frac{3\pi}{2}$

$$= 0 + 2(-1) + 3(-1) + 4(0) - 5(-1) - 6(-1)$$

$$= 0 - 2 - 3 + 0 + 5 + 6 = 6$$

Ex.3 Find all trigonometric functions of the angle made by OP with X-axis where P is (–5, 12).

Solution: Let θ be the measure of the angle in standard position whose terminal arm passes through P(–5,12).

$$r = OP = \sqrt{(-5)^2 + 12^2} = 13$$

$P(x, y) = (-5, 12)$ $\therefore$ $x = -5$, $y = 12$

$$\sin\theta = \frac{y}{r} = \frac{12}{13} \qquad \text{cosec}\,\theta = \frac{r}{y} = \frac{13}{12}$$

$$\cos\theta = \frac{x}{r} = \frac{-5}{13} \qquad \sec\theta = \frac{r}{x} = \frac{-13}{12}$$

$$\tan\theta = \frac{y}{x} = \frac{-12}{5} \qquad \cot\theta = \frac{x}{y} = \frac{-5}{12}$$

Ex.4 $\sec\theta = -3$ and $\pi < \theta < \frac{3\pi}{2}$ then find the values of other trigonometric functions.

Solution : Given $\sec\theta = -3$

$$\therefore \cos\theta = -\frac{1}{3}$$

We have $\tan^2\theta = \sec^2\theta - 1$

$\therefore$ $\tan^2\theta = 9 - 1 = 8$

$\therefore$ $\tan^2\theta = 8$ and $\pi < \theta < \frac{3\pi}{2}$, the third quadrant.

$\therefore$ $\tan\theta = 2\sqrt{2}$ Hence $\cot\theta = \frac{1}{2\sqrt{2}}$

Also we have, $\sin\theta = \tan\theta\cos\theta$

$$= 2\sqrt{2}\left(-\frac{1}{3}\right) = -\frac{2\sqrt{2}}{3}$$

$$\therefore \text{cosec}\,\theta = -\frac{3}{2\sqrt{2}}$$

Ex.5 If $\sec x = \frac{13}{5}$, x lies in the fourth quadrant, find the values of other trigonometric functions.

Solution : Since $\sec x = \frac{13}{5}$, we have $\cos x = \frac{5}{13}$

Now $\tan^2 x = \sec^2 x - 1$

$$\therefore \tan^2 x = \left(\frac{13}{5}\right)^2 - 1 = \frac{169}{25} - 1 = \frac{144}{25}$$

$\therefore \tan^2 x = \frac{144}{25}$ and x lies in the fourth quadrant.

$$\therefore \tan x = \frac{-12}{5} \qquad \cot x = -\frac{5}{12}$$

Further we have, $\sin x = \tan x \times \cos x$

$$= -\frac{12}{5} \times \frac{5}{13} = -\frac{12}{13}$$

And $\operatorname{cosec} x = \frac{1}{\sin x} = -\frac{13}{12}$

Ex.6 If $\tan A = \frac{4}{3}$, find the value of

$$\frac{2\sin A - 3\cos A}{2\sin A + 3\cos A}$$

Solution : Given expression

$$\frac{2\sin A - 3\cos A}{2\sin A + 3\cos A} = \frac{2\frac{\sin A}{\cos A} - 3\frac{\cos A}{\cos A}}{2\frac{\sin A}{\cos A} + 3\frac{\cos A}{\cos A}}$$

$$= \frac{2\tan A - 3}{2\tan A + 3}$$

$$= \frac{2\left(\frac{4}{3}\right) - 3}{2\left(\frac{4}{3}\right) + 3} = -\frac{1}{17}$$

Ex.7 If $\sec\theta = \sqrt{2}$, $\frac{3\pi}{2} < \theta < 2\pi$ then find the value of $\frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta}$.

Solution : Given $\sec\theta = \sqrt{2}$ $\therefore$ $\cos\theta = \frac{1}{\sqrt{2}}$

Now $\tan^2\theta = \sec^2\theta - 1 = 2 - 1 = 1$

$\therefore \tan^2\theta = 1$ and $\frac{3\pi}{2} < \theta < 2\pi$ (the fourth quadrant)

$\therefore \tan\theta = -1$. Hence $\cot\theta = -1$

Now $\sin\theta = \tan\theta\cos\theta = (-1)\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$

Hence $\operatorname{cosec}\theta = -\sqrt{2}$

$$\frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cos\theta-\operatorname{cosec}\theta} = \frac{1+(-1)+(-\sqrt{2})}{1+(-1)-(-\sqrt{2})} = -1$$

Ex.8 If $\sin\theta = -\frac{3}{5}$ and $180° < \theta < 270°$ then find all trigonometric functions of θ.

Solution : Since $180° < \theta < 270°$, θ lies in the third quadrant.

Since, $\sin\theta = -\frac{3}{5}$ $\therefore$ $\operatorname{cosec}\theta = -\frac{5}{3}$

Now $\cos^2\theta = 1 - \sin^2\theta$

$\therefore \cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$

$\therefore \cos\theta = -\frac{4}{5}$ $\qquad \therefore \sec\theta = -\frac{5}{4}$

Now $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\therefore \tan\theta = \frac{3}{4}$ $\therefore \cot\theta = \frac{4}{3}$

EXERCISE 2.1

1) Find the trigonometric functions of $0°$, $30°$, $45°$, $60°$, $150°$, $180°$, $210°$, $300°$, $330°$, $-30°$, $-45°$, $-60°$, $-90°$, $-120°$, $-225°$, $-240°$, $-270°$, $-315°$

2) State the signs of
   i) tan380°   ii) cot230°   iii) sec468°

3) State the signs of $\cos 4^c$ and $\cos 4°$. Which of these two functions is greater ?

4) State the quadrant in which θ lies if

i) $\sin\theta < 0$ and $\tan\theta > 0$

ii) $\cos\theta < 0$ and $\tan\theta > 0$

5) Evaluate each of the following :

i) $\sin 30° + \cos 45° + \tan 180°$

ii) $\text{cosec} 45° + \cot 45° + \tan 0°$

iii) $\sin 30° \times \cos 45° \times \tan 360°$

6) Find all trigonometric functions of angle in standard position whose terminal arm passes through point (3, –4).

7) If $\cos\theta = \frac{12}{13}$, $0 < \theta < \frac{\pi}{2}$, find the value of $\frac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta}$, $\frac{1}{\tan^2\theta}$

8) Using tables evaluate the following :

i) $4\cot 45^0 - \sec^2 60^0 + \sin 30^0$

ii) $\cos^2 0 + \cos^2\frac{\pi}{6} + \cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{2}$

9) Find the other trigonometric functions if

i) If $\cos\theta = -\frac{3}{5}$ and $180^0 < \theta < 270^0$.

ii) If $\sec A = -\frac{25}{7}$ and A lies in the second quadrant.

iii) If $\cot x = \frac{3}{4}$, x lies in the third quadrant.

iv) $\tan x = \frac{-5}{12}$, x lies in the fourth quadrant.

Let's Learn

Fundamental Identities

2.2 Fundamental Identities :

A trigonometric identity represents a relationship that is always for all admissible values in the domain. For example $\text{cosec}\theta = \frac{1}{\sin\theta}$ is true for all admissible values of θ. Hence this is an identity. Identities enable us to simplify complicated expressions. They are basic tools of trigonometry which are being used in solving trigonometric equations.

The fundamental identities of trigonometry, namely.

1) $\sin^2\theta + \cos^2\theta = 1$,

using this identity we can derive simple relations in trigonometry functions

e.g. $\cos\theta = \pm\sqrt{1-\sin^2\theta}$ and

$\sin\theta = \pm\sqrt{1-\cos^2\theta}$

2) $1 + \tan^2\theta = \sec^2\theta$, if $\theta \neq \frac{\pi}{2}$

3) $1 + \cot^2\theta = \text{cosec}^2\theta$, if $\theta \neq 0$

These relations are called fundamental identities of trigonometry.

2.2.1 Domain and Range of Trigonometric functions :

Now we will find domain and range of trigonometric functions expressed as follows.

We now study $\sin\theta$, $\cos\theta$, $\tan\theta$ as functions of real variable θ. Here θ is measured in radians.

We have defined $\sin\theta$ and $\cos\theta$, *where* θ is a real number. If α and θ are co-terminal angles and if $0° \leq \alpha \leq 360°$, then $\sin\theta = \sin\alpha$, and $\cos\theta = \cos\alpha$. Hence the domain of these function is R.

Let us find the range $\sin\theta$ and $\cos\theta$

We have, $\sin^2\theta + \cos^2\theta = 1$

i) Consider $y = \sin\theta$ where $\theta \in R$ and $y \in [-1. 1]$

The domain of sine function is R and range is [–1, 1].

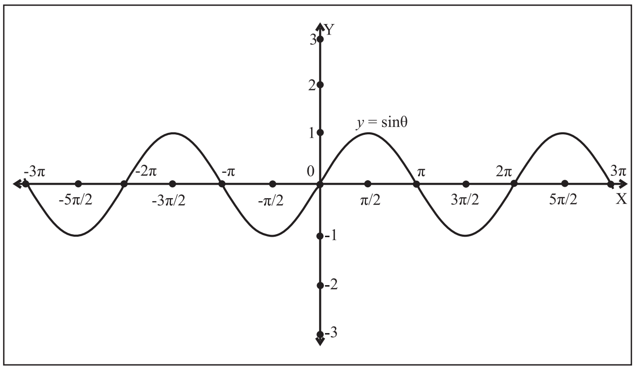

Fig. 2.15

ii) Consider $y = \cos\theta$ where $\theta \in R$ and $y \in [-1, 1]$

The domain of cosine function is R and range is [–1, 1].

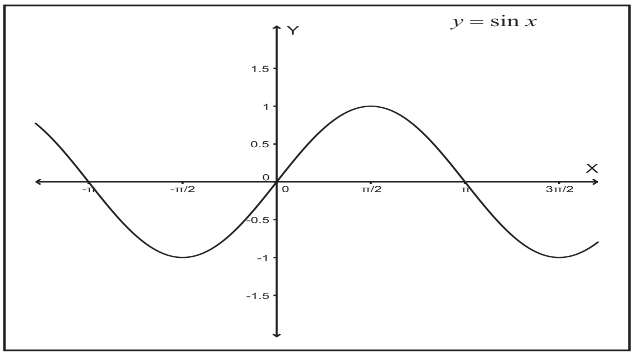

Fig. 2.16

iii) Consider $y = \tan\theta$, $\tan\theta$ does not exist for

$\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ .....

In general $\tan\theta$ does not exist if $\theta = (2n + 1)\frac{\pi}{2}$, where $n \in I$

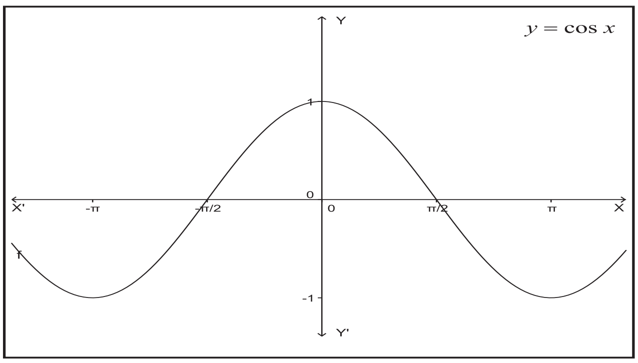

Fig. 2.17

The domain of $\tan\theta$ is R except

$\theta = (2n + 1)\frac{\pi}{2}$,

As $\theta \to \frac{\pi}{2}^-$, $\tan\theta \to +\infty$ and as $\theta \to \frac{\pi}{2}^+$, $\tan\theta \to -\infty$.

when you learn the concept of the limits you will notice.

Since $\tan\theta = \frac{y}{x}$, value of $\tan\theta$ can be any real number, range of tan function is R.

iv) Consider $y = \text{cosec}\theta$

$\text{cosec}\theta$ does not exist for $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi$ ...

In general $\text{cosec}\theta$ does not exist if $\theta = n\pi$, where $n \in I$.

The domain of $\text{cosec}\theta$ is R except $\theta = n\pi$, and range is R.

The domain of sine function is R and range is [–1, 1].

Now as $-1 \le \sin\theta \le 1$, $\text{cosec}\theta \ge 1$

or $\text{cosec}\theta \le -1$.

$\therefore$ The range of cosecant function is

$\{y \in R : |y| \ge 1\} = R - (-1, 1)$

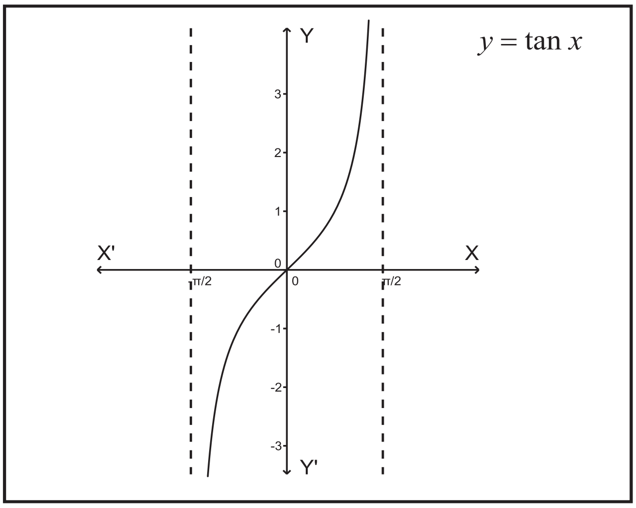

Fig. 2.18

v) Consider $y = \sec\theta$

$\sec\theta$ does not exist for $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$

In general $\sec\theta$ does not exist if

$\theta = (2n + 1)\frac{\pi}{2}$, where $n \in I$.

The domain of $\sec\theta$ is R except $\theta = (2n + 1)\frac{\pi}{2}$, and range is $R - (-1, 1)$

Now as $-1 \le \cos\theta \le 1$, $\sec\theta \ge 1$ or $\sec\theta \le -1$

$\therefore$ The range of secant function is $\{y \in R : |y| \ge 1\} = R - (-1, 1)$

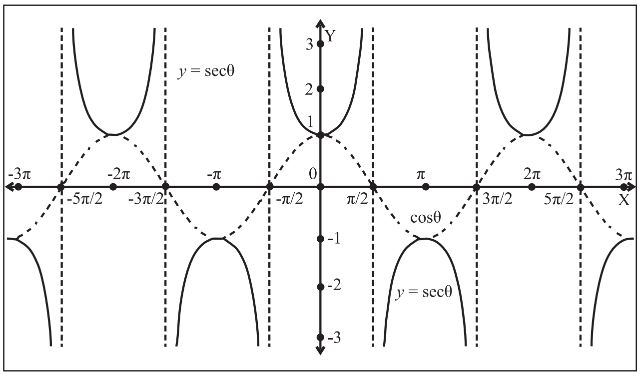

Fig. 2.19

vi) Consider $y = \cot\theta$

$\cot\theta$ does not exist for $\theta = 0, \pm\pi, \pm2\pi, \pm3\pi \ldots$

In general $\cot\theta$ does not exist if $\theta = n\pi$, where $n \in I$.

The domain of $\cot\theta$ is R except $\theta = n\pi$, and range is R.

The domain of sine function is R and range is $[-1, 1]$.

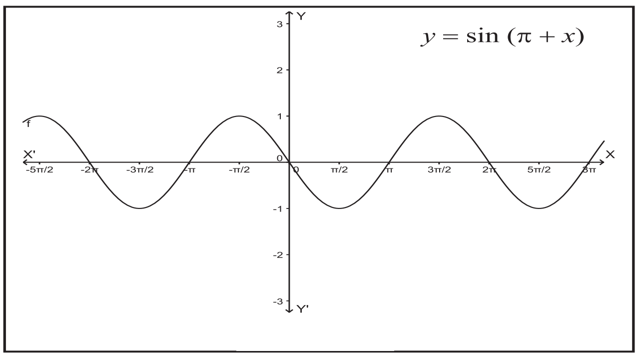

Fig. 2.20

Now as $-1 \le \cos\theta \le 1$, $\sec\theta \ge 1$ or $\sec\theta \le 1$

Similarly, $-1 \le \sin\theta \le 1$, $\text{cosec}\theta \ge 1$ or $\text{cosec}\theta \le -1$

Since $\cot\theta = \frac{y}{x}$, value of can be any real number, range of cot function is R.

2.2.2 Periodicity of Trigonometric functions:

A function is said to be a periodic function if there exists a constant p such that $f(x + p) = f(x)$ for all x in the domain.

$\therefore f(x) = f(x + p) = f(x + 2p) = \ldots = f(x - p) = f(x - 2p) = \ldots$

The smallest positive value of p which satisfies the above relation is called the fundamental period or simply the period of f.

Ex. $\sin(x + 2\pi) = \sin(x + 4\pi) = \sin x = \sin(x - 2\pi) = \sin(x - 4\pi)$

Thus sinx is a periodic function with period 2π.

Similarly cos$x$, cosec$x$ and sec$x$ are periodic functions with period 2π.

But tan$x$ and cot$x$ are periodic functions with period π. Because of $\tan(x+\pi) = \tan x$ for all x.

The following table gives the domain, range and period of trigonometric functions.

Trigono-metric functions	Domain	Range	Period
$\sin\theta$	R	$[-1, 1]$	2π
$\cos\theta$	R	$[-1, 1]$	2π
$\tan\theta$	$R - \{(2n + 1)\frac{\pi}{2} : n \in I\}$	R	π
$\text{cosec}\theta$	$R - \{n\pi : n \in I\}$	$R - (-1, 1)$	2π
$\sec\theta$	$R - \{(2n + 1)\frac{\pi}{2} : n \in I\}$	$R - (-1, 1)$	2π
$\cot\theta$	$R - \{n\pi : n \in I\}$	R	π

SOLVED EXAMPLES

Ex.1 Find the value of $\sin \frac{41\pi}{4}$.

Solution : We know that sine function is periodic with period 2π.

$$\therefore \sin \frac{41\pi}{4} = \sin\left(10\pi + \frac{\pi}{4}\right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Ex.2 Find the value of cos 765°.

Solution : We know that cosine function is periodic with period 2π.

$$\therefore \cos 765° = \cos(720° + 45°)$$
$$= \cos(2 \times 360° + 45°)$$
$$= \cos 45° = \frac{1}{\sqrt{2}}$$

Let's Learn

2.9 Graphs of trigonometric functions :

Introduction : In this section we shall study the graphs of trigonometric functions. Consider x to be a real number or measure of an angle in radian. We know that all trigonometric functions are periodic. The periods of sine and cosine functions is 2π and the period of tangent function is π. These periods are measured in radian.

(i) The graph of sine function:

Consider $y = \sin x$, for $-\pi < x < \pi$. Here x represents a variable angle. The table of values is as follows:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
y	0	0.5	0.71	0.87	1	0.87	0.71	0.5	0

Using the result $\sin(-\theta) = -\sin\theta$, we have following table:

x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0
y	0	–0.5	–0.71	–0.87	–1	–0.87	–0.71	–0.5	0

Take the horizontal axis to be the X– axis and the vertical axis to be the Y– axis.

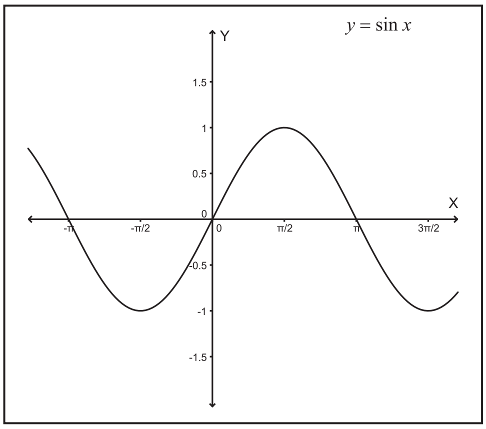

Fig. 2.21

The graph of $y = \sin x$ is shown above. Since the period of sine function is 2π It means that take the curve and shift it 2π to left or right, then the curve falls back on itself. Also note that the graph is with in one unit of the Y– axis. The graph increases and decreases periodically.

(ii) The graph of cosine function: Consider $y = \cos x$, for $-\pi < x < \pi$. Here x represents a variable angle. The table of values is as follows:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
y	1	0.87	0.71	0.5	0	0.5	0.71	0.87	1

Using the result $\cos(-\theta) = \cos\theta$, we have following table:

x	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0
Y	1	0.87	0.71	0.5	0	0.5	0.71	0.87	1

Take the horizontal axis to be the X– axis and the vertical axis to be the Y– axis.

The graph of y = cos$x$ is shown below. Since the period of cosine function is 2π. It means that take the curve and shift it 2π to left or right, then the curve falls back on itself. Also note that the graph is with in one unit of the Y– axis. The graph increases and decreases periodically.

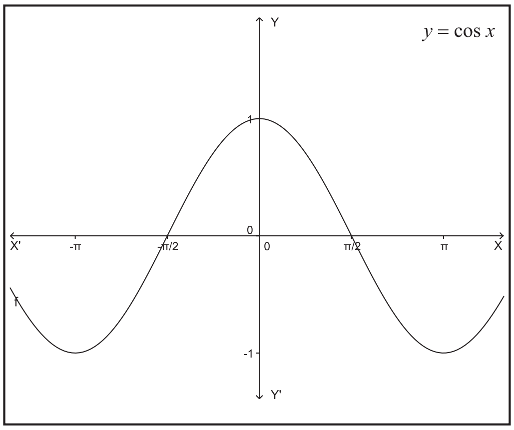

Fig. 2.22

(iii) The graph of tangent function:

Let y = tan$x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Note that does not exist for $x = \frac{\pi}{2}$. As x increases from 0 to $\frac{\pi}{2}$:

1) sin$x$ increases from 0 to 1 and

2) cos$x$ decreases from 1 to 0.

$\therefore$ tan$x$ = $\frac{\sin x}{\cos x}$ will increase indefinitely as x starting from 0 approaches to $\frac{\pi}{2}$. Similarly starting from 0 approaches to $-\frac{\pi}{2}$, tan$x$ decreases indefinitely. The corresponding values of x ad y are as in the following table:

x	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	–1.73	–1	–0.58	0	0.58	1	1.73

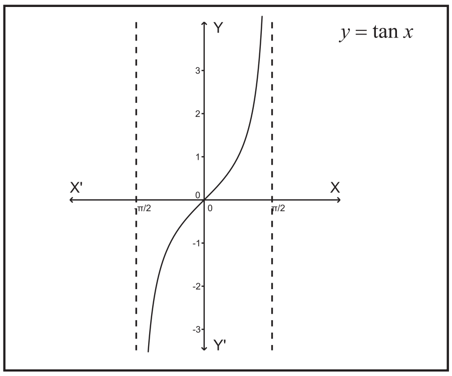

Fig. 2.23

(Activity) :

1) Use the tools in Geogebra to draw the different types of graphs of trigonometric functions.

Geogebra is an open source application available on internet.

2) Plot the graphs of cosecant, secant and cotangent functions.

SOLVED EXAMPLES

Ex. 1 If $\tan\theta + \frac{1}{\tan\theta} = 2$ then find the value of $\tan^2\theta + \frac{1}{\tan^2\theta}$

Solution : We have $\tan\theta + \frac{1}{\tan\theta} = 2$

Squaring both sides, we get

$$\tan^2\theta + 2\tan\theta \times \frac{1}{\tan\theta} + \frac{1}{\tan^2\theta} = 4$$

$$\therefore \tan^2\theta + 2 + \frac{1}{\tan^2\theta} = 4$$

$$\therefore \tan^2\theta + \frac{1}{\tan^2\theta} = 2$$

Ex. 2 Which of the following is true?

i) $2\cos^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

ii) $\frac{\cot A - \tan B}{\cot B - \tan A} = \cot A \tan B$

iii) $\frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta} = \sin\theta + \cos\theta$

Solution :

i) $2\cos^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$$\text{RHS} = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$= \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \cos^2\theta - \sin^2\theta$$

$$= \cos^2\theta - (1-\cos^2\theta)$$

$$= 2\cos^2\theta - 1 \neq \text{LHS}$$

Since the LHS ≠ RHS, given equation is not true.

ii) $\frac{\cot A - \tan B}{\cot B - \tan A} = \cot A \tan B$

Solution : Substitute $A = B = 45°$

$$\text{LHS} = \frac{\cot 45° - \tan 45°}{\cot 45° - \tan 45°}$$

$$= \frac{1-1}{1+1} = \frac{0}{1} = 0$$

RHS = $\cot 45° \tan 45° = 1$

As LHS ≠ RHS, the given equation is not true.

Note : 'One counter example is enough' to prove that a mathematical statement is wrong.

iii) $\frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta} = \sin\theta + \cos\theta$

$$\text{LHS} = \frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta}$$

$$= \frac{\cos^2\theta}{\cos\theta - \sin\theta} + \frac{\sin^2\theta}{\sin\theta - \cos\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$$

$$= \frac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta - \sin\theta}$$

$$= \sin\theta + \cos\theta = \text{RHS}$$

Since the LHS = RHS, given equation is true.

Ex.3 If $5\tan A = \sqrt{2}$, $\pi < A < \frac{3\pi}{2}$ and $\sec B = \sqrt{11}$, $\frac{3\pi}{2} < B < 2\pi$ then find the value of $\operatorname{cosec} A - \tan B$.

Solution : $5\tan A = \sqrt{2}$

$\therefore \tan A = \frac{\sqrt{2}}{5}$ and $\cot A = \frac{5}{\sqrt{2}}$

As $\operatorname{cosec}^2 A = 1 + \cot^2 A$

$$= 1 + \left(\frac{5}{\sqrt{2}}\right)^2 = \frac{27}{2}$$

$\therefore \operatorname{cosec}^2 A = \frac{27}{2}$ and $\pi < A < \frac{3\pi}{2}$ (the third quadrant)

$$\therefore \operatorname{cosec} A = -\frac{\sqrt{27}}{\sqrt{2}}$$

Now $\sec B = \sqrt{11}$

As $\tan^2 B = \sec^2 B - 1 = 10$

Thus, $\tan^2 B = 10$ and $\frac{3\pi}{2} < B < \pi$ (the fourth quadrtant)

$\therefore \tan B = -\sqrt{10}$

Now $\operatorname{cosec} A - \tan B = -\frac{\sqrt{27}}{\sqrt{2}} - (-\sqrt{10})$

$$= \frac{\sqrt{20} - \sqrt{27}}{\sqrt{2}}$$

Ex.4 If $\tan\theta = \frac{1}{\sqrt{7}}$ then evaluate $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$

Solution : Given $\tan\theta = \frac{1}{\sqrt{7}}$

$\therefore \cot\theta = \sqrt{7}$

Since, $\text{cosec}^2\theta = 1 + \cot^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$

$\therefore \text{cosec}^2\theta - \sec^2\theta = \cot^2\theta - \tan^2\theta$

$\therefore \text{cosec}^2\theta + \sec^2\theta = \cot^2\theta + \tan^2\theta + 2$

$$\frac{\text{cosec}^2\theta - \sec^2\theta}{\text{cosec}^2\theta + \sec^2\theta} = \frac{\cot^2\theta - \tan^2\theta}{\cot^2\theta + \tan^2\theta + 2}$$

$$= \frac{7 - \frac{1}{7}}{7 + \frac{1}{7} + 2} = \frac{48}{64} = \frac{3}{4}$$

Ex.5 Prove that $\cos^6\theta + \sin^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

Solution : $(a^3+b^3) = (a+b)^3 - 3ab(a+b)$

L.H.S. $= \cos^6\theta + \sin^6\theta$

$= (\cos^2\theta)^3 + (\sin^2\theta)^3$

$= (\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)$

$= 1 - 3\sin^2\theta\cos^2\theta$ (Since $\sin^2\theta + \cos^2\theta = 1$)

$=$ R.H.S.

Ex.6 Eliminate θ from the following :

(i) $x = a\cos\theta$, $y = b\sin\theta$

(ii) $x = a\cos^3\theta$, $y = b\sin^3\theta$

(iii) $x = 2 + 3\cos\theta$, $y = 5 + 3\sin\theta$

Solution :

(i) $x = a\cos\theta$, $y = b\sin\theta$

$\therefore \cos\theta = \frac{x}{a}$ and $\sin\theta = \frac{y}{b}$

On squaring and adding, we get

$$\cos^2\theta + \sin^2\theta = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

but $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(ii) $x = a\cos^3\theta$, $y = b\sin^3\theta$

$\therefore \cos^3\theta = \frac{x}{a}$ and $\sin^3\theta = \frac{y}{b}$

$\therefore \cos\theta = \left(\frac{x}{a}\right)^{\frac{1}{3}}$ and $\sin\theta = \left(\frac{y}{b}\right)^{\frac{1}{3}}$

On squaring and adding, we get

$$\cos^2\theta + \sin^2\theta = \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}}$$

but $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} = 1$$

(iii) $x = 2 + 3\cos\theta$, $y = 5 + 3\sin\theta$

$x - 2 = 3\cos\theta$, $y - 5 = 3\sin\theta$

$$\cos\theta = \left(\frac{x-2}{3}\right), \quad \sin\theta = \left(\frac{y-5}{3}\right)$$

We know that,

$\cos^2\theta + \sin^2\theta = 1$

Therefore,

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-5}{3}\right)^2 = 1$$

$\therefore (x-2)^2 + (y-5)^2 = (3)^2$

$\therefore (x-2)^2 + (y-5)^2 = 9$

Ex.7 If $2\sin^2\theta + 7\cos\theta = 5$ then find the permissible values of $\cos\theta$.

Solution : We know that $\sin^2\theta = 1 - \cos^2\theta$

Given equation $2\sin^2\theta + 7\cos\theta = 5$ becomes

$2(1 - \cos^2\theta) + 7\cos\theta = 5$

$\therefore 2 - 2\cos^2\theta + 7\cos\theta - 5 = 0$

$\therefore 2\cos^2\theta - 7\cos\theta + 3 = 0$

$\therefore 2\cos^2\theta - 6\cos\theta - \cos\theta + 3 = 0$

$\therefore (2\cos\theta - 1)(\cos\theta - 3) = 0$

$\therefore \cos\theta = 3$ or $\cos\theta = \frac{1}{2}$

But $\cos\theta$ cannot be greater than 1

$\therefore$ Permissible value of $\cos\theta$ is $\frac{1}{2}$.

Ex. 8 Solve for θ, if $4\sin^2\theta - 2(\sqrt{3}+1)\sin\theta + \sqrt{3} = 0$

Solution : $4\sin^2\theta - 2(\sqrt{3}+1)\sin\theta + \sqrt{3} = 0$ is a quadratic equation in $\sin\theta$. Its roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 4$, $b = -2(\sqrt{3}+1)$, $c = \sqrt{3}$

$$\therefore \sin\theta = \frac{2(\sqrt{3}+1) \pm 2\sqrt{\left[2(\sqrt{3}+1)\right]^2 - 4(4)(\sqrt{3})}}{2(4)}$$

$$= \frac{2(\sqrt{3}+1) \pm 2\sqrt{\left[(\sqrt{3}+1)\right]^2 - (4)(\sqrt{3})}}{2(4)}$$

$$= \frac{(\sqrt{3}+1) \pm \sqrt{\left[(\sqrt{3}+1)\right]^2 - (4)(\sqrt{3})}}{(4)}$$

$$= \frac{(\sqrt{3}+1) \pm \sqrt{3 + 2\sqrt{3} + 1 - 4(\sqrt{3})}}{4}$$

$$= \frac{(\sqrt{3}+1) \pm \sqrt{4 - 2(\sqrt{3})}}{4}$$

$$= \frac{(\sqrt{3}+1) \pm \sqrt{\left[\sqrt{3}-1\right]^2}}{4}$$

$$= \frac{(\sqrt{3}+1) \pm (\sqrt{3}-1)}{4}$$

$$= \frac{\sqrt{3}}{2} \text{ or } \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{6} \text{ or } \frac{\pi}{3}$$

Ex. 9 If $\tan\theta + \sec\theta = 1.5$ then find $\tan\theta$, $\sin\theta$ and $\sec\theta$.

Solution : Given $\tan\theta + \sec\theta = 1.5$

$$\therefore \tan\theta + \sec\theta = \frac{3}{2} \quad \ldots (1)$$

Now $\sec^2\theta - \tan^2\theta = 1$

$\therefore (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

$\therefore \frac{3}{2}(\sec\theta - \tan\theta) = 1$

$\therefore (\sec\theta - \tan\theta) = \frac{2}{3} \quad \ldots (2)$

From (1) and (2), we get, $2\sec\theta = \frac{13}{6}$

$\therefore \sec\theta = \frac{13}{12}$ and $\cos\theta = \frac{12}{13}$

$\therefore \tan\theta = \frac{5}{12}$ and $\sin\theta = \frac{5}{13}$

Ex. 10 Prove that

$$\frac{\sin\theta}{1-\cos\theta} + \frac{\tan\theta}{1+\cos\theta} = \sec\theta\,\mathrm{cosec}\theta + \cot\theta$$

Solution : LHS $= \frac{\sin\theta}{1-\cos\theta} + \frac{\tan\theta}{1+\cos\theta}$

$$= \frac{\sin\theta(1+\cos\theta) + \tan\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{\sin\theta + \sin\theta\cos\theta + \tan\theta - \tan\theta\cos\theta}{1-\cos^2\theta}$$

$$= \frac{\sin\theta + \sin\theta\cos\theta + \tan\theta - \tan\theta\cos\theta}{\sin^2\theta}$$

$$= \frac{\sin\theta}{\sin^2\theta} + \frac{\sin\theta\cos\theta}{\sin^2\theta} + \frac{\tan\theta}{\sin^2\theta} - \frac{\tan\theta\cos\theta}{\sin^2\theta}$$

$$= \frac{\sin\theta}{\sin^2\theta} + \frac{\sin\theta\cos\theta}{\sin^2\theta} + \frac{\tan\theta}{\sin^2\theta} - \frac{\sin\theta}{\sin^2\theta}$$

$$= \frac{\sin\theta\cos\theta}{\sin^2\theta} + \frac{\tan\theta}{\sin^2\theta}$$

$$= \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta\cos\theta} = \cot\theta + \mathrm{cosec}\theta\ \sec\theta$$

$= \sec\theta\,\mathrm{cosec}\theta + \cot\theta = \text{RHS}$

Ex. 11 Prove that

$$\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$$

Solution : LHS = $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}$

$$= \frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} \times \frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}$$

$$= \frac{(\sec\theta - \tan\theta)^2}{\sec^2\theta - \tan^2\theta}$$

$$= \frac{\sec^2\theta + \tan^2\theta - 2\sec\theta\tan\theta}{1}$$

$= 1 + \tan^2\theta + \tan^2\theta - 2\sec\theta\tan\theta$

$= 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$ = RHS

Ex.12 Prove that $(\sec A - \tan A)^2 = \dfrac{1-\sin A}{1+\sin A}$

Solution : LHS = $(\sec A - \tan A)^2$

$= \sec^2 A + \tan^2 A - 2\sec A\tan A$

$$= \frac{1}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A} - 2\frac{\sin A}{\cos A\cos A}$$

$$= \frac{1+\sin^2 A - 2\sin A}{\cos^2 A}$$

$$= \frac{(1-\sin A)^2}{1-\sin^2 A} = \frac{1-\sin A}{1+\sin A} = \text{RHS}$$

Let's Learn

2.2.4 Polar Co-ordinate system : Consider O as the origin and OX as X-axis. P (x,y) is any point in the plane. Let OP = r and m∠XOP = θ. Then the ordered pair (r, θ) determines the position of point P. Here (r, θ) are called the polar coordinates of P. The fixed point O is called the Pole and the fixed ray OX or X-axis is called as the polar axis.

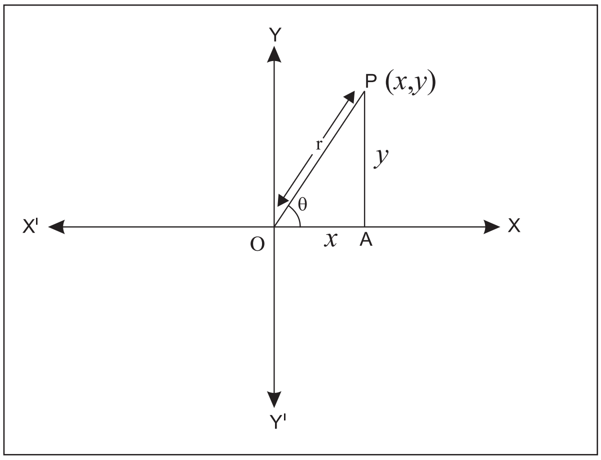

Fig. 2.24

The Cartesian co-ordinates of the point P(r, θ) will be given by relations :

$x = r\cos\theta$ and $y = r\sin\theta$

From these relations we get

$r = \sqrt{x^2+y^2}$ and $\tan\theta = \dfrac{y}{x}$

SOLVED EXAMPLE

Ex. Find the polar co-ordinates of the point whose Cartesian coordinates are (3,3).

Solution : Here $x = 3$ and $y = 3$

To find r and θ.

$r = \sqrt{x^2+y^2}$ and $\tan\theta = \dfrac{y}{x}$

$r = \sqrt{x^2+y^2} = \sqrt{3^2+3^2} = \sqrt{18} = 3\sqrt{2}$

$\therefore r = 3\sqrt{2}$

Since point P lies in the first quadrant,θ is an angle in the first quadrant.

$\tan\theta = \dfrac{y}{x} = \dfrac{3}{3} = 1 \quad \therefore \theta = 45°$

Polar co-ordinates of P are (r, θ) =(1, 45°)

EXERCISE 2.2

1) If $2\sin A = 1 = \sqrt{2}\cos B$ and $\frac{\pi}{2} < A < \pi$, $\frac{3\pi}{2} < B < 2\pi$, then find the value of $\frac{\tan A + \tan B}{\cos A - \cos B}$

2) If $\frac{\sin A}{3} = \frac{\sin B}{4} = \frac{1}{5}$ and A, B are angles in the second quadrant then prove that $4\cos A + 3\cos B = -5$.

3) If $\tan\theta = \frac{1}{2}$, evaluate $\frac{2\sin\theta + 3\cos\theta}{4\cos\theta + 3\sin\theta}$

4) Eliminate θ from the following :

i) $x = 3\sec\theta$, $y = 4\tan\theta$

ii) $x = 6\operatorname{cosec}\theta$, $y = 8\cot\theta$

iii) $x = 4\cos\theta - 5\sin\theta$, $y = 4\sin\theta + 5\cos\theta$

iv) $x = 5 + 6\operatorname{cosec}\theta$, $y = 3 + 8\cot\theta$

v) $2x = 3 - 4\tan\theta$, $3y = 5 + 3\sec\theta$

5) If $2\sin^2\theta + 3\sin\theta = 0$, find the permissible values of $\cos\theta$.

6) If $2\cos^2\theta - 11\cos\theta + 5 = 0$ then find possible values of $\cos\theta$.

7) Find the acute angle θ such that $2\cos^2\theta = 3\sin\theta$

8) Find the acute angle θ such that $5\tan^2\theta + 3 = 9\sec\theta$

9) Find $\sin\theta$ such that $3\cos\theta + 4\sin\theta = 4$

10) If $\operatorname{cosec}\theta + \cot\theta = 5$, then evaluate $\sec\theta$.

11) If $\cot\theta = \frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{4}$ then find the value of $4\operatorname{cosec}\theta + 5\cos\theta$.

12) Find the Cartesian co-ordinates of points whose polar coordinates are :

i) (3, 90°) ii) (1,180°)

13) Find the polar co-ordinates of points whose Cartesian co-ordinates are :

i) (5,5) ii) $(1, \sqrt{3})$ iii) (–1, –1)

iv) $(-\sqrt{3}, 1)$

14) Find the value of

i) $\sin\frac{19\pi^c}{3}$ ii) cos 1140° iii) $\cot\frac{25\pi^c}{3}$

15) Prove the following identities:

i) $(1 + \tan^2 A) + \left(1 + \frac{1}{\tan^2 A}\right) = \frac{1}{\sin^2 A - \sin^4 A}$

ii) $(\cos^2 A - 1)(\cot^2 A + 1) = -1$

iii) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2 = (1 + \operatorname{cosec}\theta\sec\theta)^2$

iv) $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2$

v) $\frac{\tan^3\theta}{1+\tan^2\theta} + \frac{\cot^3\theta}{1+\cot^2\theta} = \sec\theta\ \text{cose}\theta - 2\sin\theta\cos\theta$

vi) $\frac{1}{\sec\theta + \tan\theta} - \frac{1}{\cos\theta} = \frac{1}{\cos\theta} - \frac{1}{\sec\theta - \tan\theta}$

vii) $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$

viii) $\frac{\tan\theta}{\sec\theta - 1} = \frac{\sec\theta + 1}{\tan\theta}$

ix) $\frac{\cot\theta}{\operatorname{cosec}\theta - 1} = \frac{\operatorname{cosec}\theta + 1}{\cot\theta}$

x) $(\sec A + \cos A)(\sec A - \cos A) = \tan^2 A + \sin^2 A$

xi) $1 + 3\operatorname{cosec}^2\theta \cdot \cot^2\theta + \cot^6\theta = \operatorname{cosec}^6\theta$

xii) $\frac{1 - \sec\theta + \tan\theta}{1 + \sec\theta - \tan\theta} = \frac{\sec\theta + \tan\theta - 1}{\sec\theta + \tan\theta + 1}$

Let's Remember

1)

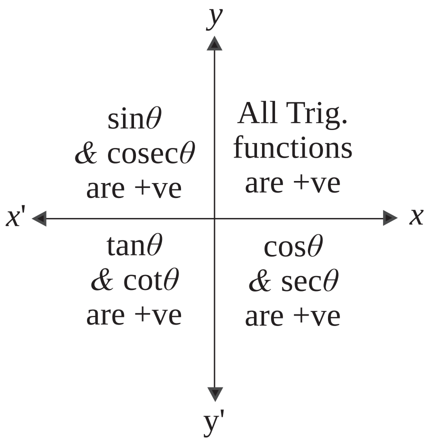

Fig. 2.25

2) All trigonometric functions are positive for θ in the first quadrant.

3) Only $\sin\theta$ is positive; $\cos\theta$ and $\tan\theta$ are negative for θ in the second quadrant.

4) Only $\tan\theta$ is positive $\sin\theta$ and $\cos\theta$ are negative for θ in the third quadrant.

5) Only $\cos\theta$ is positive; $\sin\theta$ and $\tan\theta$ are negative for θ in the fourth quadrant.

6) Signs of $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$ are same as signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ respectively.

7) The fundamental identities of trigonometric functions.

1) $\sin^2\theta + \cos^2\theta = 1$

2) $1 + \tan^2\theta = \sec^2\theta$, If $\theta \neq \frac{\pi}{2}$

3) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$, if $\theta \neq 0$

8) Domain, Range and Periodicity of Trigonometric functions

Trigono-metric func-tions	Domain	Range	Period
$\sin\theta$	R	[–1,1]	2π
$\cos\theta$	R	[–1,1]	2π
$\tan\theta$	$R - \{(2n+1)\frac{\pi}{2} : n \in I\}$	R	π
$\operatorname{cosec}\theta$	$R - \{n\pi : n \in I\}$	R – (–1,1)	2π
$\sec\theta$	$R - \{(2n+1)\frac{\pi}{2} : n \in I\}$	R – (–1,1)	2π
$\cot\theta$	$R - \{n\pi : n \in I\}$	R	π

9) **Polar Co-ordinate system :** The Cartesian co-ordinates of the point P(r, θ) are given by the relations :

$x = r\cos\theta$ and $y = r\sin\theta$

where, $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$

MISCELLANEOUS EXERCISE - 2

I) Select the correct option from the given alternatives.

1) The value of the expression cos1°. cos2°. cos3°. .... cos179° =

A) –1 B) 0 C) $\frac{1}{\sqrt{2}}$ D) 1

2) $\frac{\tan A}{1+\sec A} + \frac{1+\sec A}{\tan A}$ is equal to

A) $2\operatorname{cosec} A$ B) $2\sec A$

C) $2\sin A$ D) $2\cos A$

3) If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$, $\frac{\pi}{2} < \alpha < \pi$ then sin 2 α is equal to :

A) $-\frac{24}{25}$ B) $-\frac{13}{18}$ C) $\frac{13}{18}$ D) $\frac{24}{25}$

4) If θ = 60°, then $\frac{1+\tan^2\theta}{2\tan\theta}$ is equal to

A) $\frac{\sqrt{3}}{2}$ B) $\frac{2}{\sqrt{3}}$ C) $\frac{1}{\sqrt{3}}$ D) $\sqrt{3}$

5) If secθ = m and tanθ = n, then $\frac{1}{m}\left\{(m+n)+\frac{1}{(m+n)}\right\}$ is equal to

A) 2 B) mn C) $2m$ D) $2n$

6) If cosec θ + cot θ = $\frac{5}{2}$, then the value of tan θ is

A) $\frac{14}{25}$ B) $\frac{20}{21}$ C) $\frac{21}{20}$ D) $\frac{15}{16}$

7) $1-\frac{\sin^2\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}-\frac{\sin\theta}{1-\cos\theta}$ equals

A) 0 B) 1 C) sin θ D) cos θ

8) If cosecθ – cotθ = q, then the value of cot θ is

A) $\frac{2q}{1+q^2}$ B) $\frac{2q}{1-q^2}$ C) $\frac{1-q^2}{2q}$ D) $\frac{1+q^2}{2q}$

9) The cotangent of the angles $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{6}$ are in

A) A.P. B) G.P.

C) H.P. D) Not in progression

10) The value of tan1°.tan2°tan3°..... tan89° is equal to

A) –1 B) 1 C) $\frac{\pi}{2}$ D) 2

II) Answer the following.

1) Find the trigonometric functions of :

90°, 120°, 225°, 240°, 270°, 315°, –120°, –150°, –180°, –210°, –300°, –330°

2) State the signs of

i) cosec520° ii) cot 1899° iii) sin 986°

3) State the quadrant in which θ lies if

i) tanθ < 0 and secθ >0

ii) sinθ < 0 and cosθ <0

iii) sinθ > 0 and tanθ <0

4) Which is greater sin(1856°) or sin(2006°) ?

5) Which of the following is positive ? sin(–310°) or sin(310°)

6) Show that 1– 2sinθ cosθ ≥ 0 for all θ ∈ R

7) Show that tan²θ + cot²θ ≥ 2 for all θ ∈ R

8) If sinθ = $\frac{x^2-y^2}{x^2+y^2}$ then find the values of cosθ, tanθ in terms of x and y.

9) If secθ = $\sqrt{2}$ and $\frac{3\pi}{2}<\theta<2\pi$ then evaluate

$$\frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta}$$

10) Prove the following:

i) $\sin^2 A\cos^2 B+\cos^2 A\sin^2 B+\cos^2 A\cos^2 B+\sin^2 A\sin^2 B=1$

ii) $\frac{(1+\cot\theta+\tan\theta)(\sin\theta-\cos\theta)}{\sec^3\theta-\operatorname{cosec}^3\theta}=\sin^2\theta\cos^2\theta$

iii) $\left(\tan\theta+\frac{1}{\cos\theta}\right)^2+\left(\tan\theta-\frac{1}{\cos\theta}\right)^2=2\left(\frac{1+\sin^2\theta}{1-\sin^2\theta}\right)$

iv) $2\sec^2\theta-\sec^4\theta-2\operatorname{cosec}^2\theta+\operatorname{cosec}^4\theta=\cot^4\theta-\tan^4\theta$

v) $\sin^4\theta+\cos^4\theta=1-2\sin^2\theta\cos^2\theta$

vi) $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1=0$

vii) $\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$

viii) $\sin^4\theta + 2\sin^2\theta \cdot \cos^2\theta = 1 - \cos^4\theta$

ix) $\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} + \frac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} = 2$

x) $\tan^2\theta - \sin^2\theta = \sin^4\theta \sec^2\theta$

xi) $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = \tan^2\theta + \cot^2\theta + 7$

xii) $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta \cos^2\theta)$

xiii) $\sin^6 A + \cos^6 A = 1 - 3\sin^2 A + 3\sin^4 A$

xiv) $(1 + \tan A \cdot \tan B)^2 + (\tan A - \tan B)^2 = \sec^2 A \cdot \sec^2 B$

xv) $\frac{1 + \cot\theta + \operatorname{cosec}\theta}{1 - \cot\theta + \operatorname{cosec}\theta} = \frac{\operatorname{cosec}\theta + \cot\theta - 1}{\cot\theta - \operatorname{cosec}\theta + 1}$

xvi) $\frac{\tan\theta + \sec\theta - 1}{\tan\theta + \sec\theta + 1} = \frac{\tan\theta}{\sec\theta + 1}$

xvii) $\frac{\operatorname{cosec}\theta + \cot\theta - 1}{\operatorname{cosec}\theta + \cot\theta + 1} = \frac{1 - \sin\theta}{\cos\theta}$

xviii) $\frac{\operatorname{cosec}\theta + \cot\theta + 1}{\cot\theta + \operatorname{cosec}\theta - 1} = \frac{\cot\theta}{\operatorname{cosec}\theta - 1}$